Worksheet for September 10

Problems marked with an asterisk are to be placed in your math diary.

One of the most important theorems this semester is the theorem which states if $f(x,y)$ has the property that $f_x(x,y)$ and $f_y(x,y)$ exist and are continuous in an open disk about (a,b), then $f(x,y)$ is differentiable at all points in that disk, including the point (a,b). The problems below illustrate this.

(1.) Estimate the change in the values of $f(x,y) = 6x^2y - 4xy3 + 1$, from $f(-1,3)$ to $f(-1.001, 2.04)$.

(2*). For $f(x,y) = \begin{cases} \frac{y^3}{x^2+y^2}, \text{ if } (x,y) \neq (0,0) \\ 0, \text{ if } (x,y) = (0,0), \end{cases}$ show that:

(i) $f(x,y)$ is continuous at (0,0), and thus continuous throughout $\mathbb{R}^2$.
(ii) Both partial derivatives exist at (0,0) and but are not continuous at (0,0).
(iii) Show directly that $f(x,y)$ is not differentiable at (0,0).

(3.*) For $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, \text{ if } (x,y) \neq (0,0) \\ 0, \text{ if } (x,y) = (0,0), \end{cases}$ show that:

(i) The partials of $f(x,y)$ are continuous in any open disk about (0,0), and thus, in particular, $f(x,y)$ is differentiable at (0,0).
(ii) Show directly that $f(x,y)$ is continuous at (0,0).

Optional Bonus problem. In Calculus I, $f(x)$ is differentiable at $x = a$ if $\lim_{x\to a} \frac{f(x)-f(x)}{x-a}$ exists, and we call this limit $f'(a)$. As we saw in class, this is equivalent to saying there exists a constant $f'(a) \in \mathbb{R}$ such that

$$\lim_{x\to a} \frac{f(x) - L(x)}{x - a} = 0,$$

where $L(x) = f'(a)(x-a) + f(a)$. However, for $f(x,y)$ a function of two variables, the definition for $f(x,y)$ to be differentiable at (a,b) requires

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{||(x,y) - (a,b)||} = 0,$$

for the designated $L(x,y)$. Thus, in the single variable case the denominator in the limit is just a difference, whereas in the two variable case, the denominator is a distance. The purpose of this problem is to show that we could use a distance in the Calculus I definition. In other words, show that $f(x)$ is differentiable at $x = a$ if and only if there exists a constant $f'(a) \in \mathbb{R}$ such that

$$\lim_{x\to a} \frac{f(x) - L(x)}{|x - a|} = 0, \qquad (*)$$

where $L(x) = f'(a)(x-a) + f(a)$.

This problem is worth 3 points and is to be turned at the start of the discussion session on Thursday September 12.